Corrigé examen analyse 2 bis

exo 1 :

(a) on sait que $\lim_{n\to+\infty} \left(-\frac{2}{3}\right)^n = 0$ car $\left|-\frac{2}{3}\right| < 1$

et $\lim_{n\to+\infty} \frac{3n^3+n^2+2}{n^3+n+1} = 3$ car $\frac{3n^3+n^2+2}{n^3+n+1} = \frac{\cancel{n^3}\,3(1+\overset{\nearrow 0}{1/3n}+\overset{\nearrow 0}{1/3n^3})}{\cancel{n^3}(1+\underset{\to 0}{1/n^2}+\underset{\to 0}{1/n^3})}$

donc $\lim_{n\to+\infty} u_n = 3$

(b) on a : $\frac{v_{n+1}}{v_n} = \frac{\ln(n+1)}{3^{n+1}} \times \frac{3^n}{\ln(n)} = \frac{\ln(n(1+1/n))}{\ln(n)} \times \frac{1}{3}$

$$= \frac{\ln(n)+\ln(1+1/n)}{\ln(n)} \times \frac{1}{3}$$

$$= \left(1+\frac{\overset{\nearrow 0}{\ln(1+1/n)}}{\ln n}\right)\times\frac{1}{3} \xrightarrow[n\to+\infty]{} \frac{1}{3} < 1$$

donc par le critère de d'Alembert la série de terme général v_n est convergente.

exo 2 :

(a) $f(x) = \sin 2x$ $\quad f(0) = 0$

$f'(x) = 2\cos 2x$ $\quad f'(0) = 2$

$f''(x) = -4\sin 2x$ $\quad f''(0) = 0$

$f^{(3)}(x) = -8\cos 2x$ $\quad f^{(3)}(0) = -8$

$f(x) = f(0) + x f'(0) + \frac{x^2}{2} f''(0) + \frac{x^3}{6} f^{(3)}(0) + x^3\varepsilon(x)$ avec $\varepsilon(x) \xrightarrow[x\to 0]{} 0$

$$= 2x - \frac{4}{3}x^3 + x^3\varepsilon(x)$$

(b) $g(x) = \sqrt{1-x}$ $\quad g(0) = 1$

$g'(x) = \frac{-1}{2\sqrt{1-x}}$ $\quad g'(0) = -\frac{1}{2}$

$g''(x) = -\frac{1}{4\sqrt{1-x}\,(1-x)}$ $\quad g''(0) = -\frac{1}{4}$

$g(x) = g(0) + x g'(0) + \frac{x^2}{2} g''(0) + x^2\varepsilon(x)$ avec $\varepsilon(x) \xrightarrow[x\to 0]{} 0$

$$= 1 - \frac{x}{2} - \frac{x^2}{8} + x^2\varepsilon(x)$$

exercice 3 :

(a) $$\int_0^{\pi/2} x \sin x \, dx = \int_0^{\pi/2} f(x) g'(x) \, dx = \Big[f(x) g(x) \Big]_0^{\pi/2} - \int_0^{\pi/2} f'(x) g(x) \, dx$$

$f(x) = x \quad f'(x) = 1$

$g'(x) = \sin x \quad g(x) = -\cos x$

$$\int_0^{\pi/2} x \sin x \, dx = \Big[-x \cos x \Big]_0^{\pi/2} + \int_0^{\pi/2} \cos x \, dx$$

$$= \Big[\sin x \Big]_0^{\pi/2}$$

$$= 1$$

(b) $$\int_0^1 \frac{x}{\sqrt{1+x^2}} dx = \frac{1}{2} \int_0^1 \frac{2x}{\sqrt{1+x^2}} dx = \frac{1}{2} \int_0^1 \frac{u'(x)}{\sqrt{1+u(x)^2}} dx$$

avec $u(x) = x^2$

$u'(x) = 2x$

$$\int_0^1 \frac{x}{\sqrt{1+x^2}} dx = \frac{1}{2} \int_0^1 \frac{du}{\sqrt{1+u}} = \frac{1}{2} \Big[2 \times \sqrt{1+u} \Big]_0^1$$

$$= \sqrt{2} - 1$$

exercice 4 :

$f(x,y) = (\ln x) y^2$

$\dfrac{\partial f}{\partial x}(x,y) = \dfrac{1}{x} y^2 \qquad \dfrac{\partial f}{\partial y}(x,y) = 2(\ln x) y$

$\dfrac{\partial^2 f}{\partial x^2}(x,y) = -\dfrac{1}{x^2} y^2 \qquad \dfrac{\partial^2 f}{\partial y^2}(x,y) = 2(\ln x)$

$\dfrac{\partial^2 f}{\partial x \partial y}(x,y) = \dfrac{\partial^2 f}{\partial y \partial x}(x,y) = \dfrac{2y}{x}$ (par le thm de Schwartz) exercice 5 :

$f(x,y) = 2(x+1)^2 + 3(y-2)^2$

(a) $\frac{\partial f}{\partial x}(x,y) = 4(x+1)$

$\frac{\partial f}{\partial y}(x,y) = 6(y-2)$

(b) $\frac{\partial f}{\partial x}(x_0,y_0) = 0 \iff 4(x_0+1) = 0 \iff x_0 = -1$

$\frac{\partial f}{\partial y}(x_0,y_0) = 0 \iff 6(y_0-2) = 0 \iff y_0 = 2$

Donc f admet un seul point critique en $(x_0,y_0) = (-1,2)$

(c) On a : $f(x_0,y_0) = 2(-1+1)^2 + 3(2-2)^2 = 0$

et $f(x,y) \geq 0 \quad \forall (x,y) \in \mathbb{R}^2$

$f(x,y) > 0$ pour $(x,y) \neq (x_0,y_0)$

donc (x_0,y_0) est un minimum global

C'est le seul extremum de la fonction f.